2 Geometric interpretation of complex numbers

2.1 Definition

I will start finally with a precise definition, assuming that such mathematical object as vector space $\mathbf{R}^2$ is well familiar to the students. Recall that any element of $\mathbf{R}^2$ is a (column) vector

$$\begin{bmatrix} x \\ y \end{bmatrix},$$

where x, y are just real numbers.

Definition 2.1. *The set of complex numbers, usually denoted as* $\mathbf{C}$ *(another standard notation is* $\mathbb{C}$*, but I will stick to the former), is, by definition, the vector space* $\mathbf{R}^2$*, i.e., the set of pairs of real numbers, with operations of addition and multiplication defined as follows for any* $z_1, z_2 \in \mathbf{C}$*:*

$$z_1 + z_2 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix},$$

$$z_1 \cdot z_2 = z_1 z_2 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + y_1 x_2 \end{bmatrix}.$$

The motivation to define the multiplication exactly as it is written above comes from "naive" manipulations with numbers containing square root of -1 to find the roots of a cubic polynomial, recall the previous lecture.

Recall that the standard basis of $\mathbf{R}^2$ is $e_1 = [1\ 0]^\top, e_2 = [0\ 1]^\top$, which literally means that any vector from $\mathbf{R}^2$ (and hence from $\mathbf{C}$) can be represented in a unique way as a linear combination

$$z = \begin{bmatrix} x \\ y \end{bmatrix} = xe_1 + ye_2.$$

Now note that using the multiplication defined above for any z I have $e_1 z = z e_1 = z$, so with a slight abuse of notation I will denote e_1 as 1 emphasizing that it is my unit. Moreover,

$$e_2 \cdot e_2 = e_2^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -e_1 = -1,$$

therefore, if I introduce the notation $\mathrm{i} = e_2$, then the last line reads (familiar, isn't it?)

$$\mathrm{i}^2 = -1,$$

and hence the usual expression

$$z = x + \mathrm{i}y$$

can (and should) be understood as a representation of $z \in \mathbf{C}$ with respect to the standard basis $\{1, \mathrm{i}\}$. No "imaginary" quantities anymore! Anyway, using symbol i saves so much space and time (instead of writing vectors) and makes the computations so much easier that we will always use this notation,

but a student should always remember that behind it there are always a couple of our very familiar vectors that form the standard basis of $\mathbf{R}^2$.

The important thing is that introduced in the way how it is written above operations make the set $\mathbf{C}$ into *a field* (see the textbook or any other source if this term is unfamiliar). I will leave all the (rigorous and boring) details to the textbook, see Section 1.1, and only will show how to divide complex numbers (subtraction should be absolutely obvious). For this for a complex $z = x + \mathrm{i}y$ I introduce its *conjugate* $\bar{z} = x - \mathrm{i}y$ and note that

$$z\bar{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$$

is a real number (which rigorously, again, should be understood as a vector in $\mathbf{R}^2$ with the second coordinate zero, or even better as $(x^2 + y^2)e_1$).

Now if I want to divide two complex numbers $z_1 = x_1 + \mathrm{i}y_1, z_2 = x_2 + \mathrm{i}y_2$ I do the following:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{1}{x_2^2 + y_2^2} z_1\bar{z}_2 = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \mathrm{i}\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} = a + \mathrm{i}b\,,$$

which is defined if and only if $x_2^2 + y_2^2 \neq 0$, i.e., if and only if $z_2 \neq 0$ (i.e., it is not a zero vector).

Since I have a division it means that for any nonzero $z \neq 0$ I can find its *inverse* z^{-1}, which is defined as such a complex number as $zz^{-1} = z^{-1}z = 1$. Indeed, to find an inverse means divide 1 by z, which leads to

$$z^{-1} = \frac{1}{z} = \frac{z}{z\bar{z}} = \frac{x}{x^2 + y^2} + \mathrm{i}\frac{-y}{x^2 + y^2}\,,$$

which also prove the uniqueness of the inverse.

Having defined the inverse, I can easily check now all the axioms of a field, in particular the associativity of addition and multiplication, the commutativity of addition and multiplication, distributivity, etc, please check the textbook for exact details. What is important here is that since $\mathbf{R}$ is a field and $\mathbf{C}$ is a field, *all the algebraic formulas we got used to while manipulating expressions with real numbers stays the same if real numbers replaced with complex ones.* I.e.,

$$(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$$

is true for any complex z_1, z_2.

Before finally turning to the geometric interpretation of complex numbers I would like to state as an exercise the properties of conjugate numbers:

Problem 2.1. Show that for any $z, w \in \mathbf{C}$

$$\begin{aligned}
\overline{z \pm w} &= \bar{z} \pm \bar{w},\\
\overline{zw} &= \bar{z}\,\bar{w},\\
\overline{\left(\frac{z}{w}\right)} &= \frac{\bar{z}}{\bar{w}},\\
\bar{\bar{z}} &= z,\\
\operatorname{Re} z &= \frac{z + \bar{z}}{2}\,,\\
\operatorname{Im} z &= \frac{z - \bar{z}}{2}\,,\\
z \in \mathbf{R} &\iff z = \bar{z}.
\end{aligned}$$

Here I introduced new notation $\operatorname{Re} z$ and $\operatorname{Im} z$ for the real and imaginary parts of the complex number $z = x + \mathrm{i}y$, which are defined as $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ respectively. Also, a careful student should understand now that when I write something like $z \in \mathbf{R}$ for a complex z, I again abuse the notation and mean that $z = ae_1$, for a real constant a. But this abuse does not lead to any mistakes, everyone does it, and so I will be doing the same thing (to be a little more formal: Strictly speaking the set of real numbers is *not* a subset of complex numbers $\mathbf{C}$, which are *pairs* of real numbers, but the field $\mathbf{R}$ is isomorphic to a subfield of $\mathbf{C}$, which is the set of all vectors with the second coordinate zero; if the last sentence does not make much sense to you, do not worry, we won't need it really).

Problem 2.2. Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,$$

be a polynomial of complex variable z. $w \in \mathbf{C}$ is by definition its root if $p(w) = 0$. Show that if all the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real and if w is a root then p has another root $\bar{w}$.

Problem 2.3. Show that for any real matrix $\boldsymbol{A}$ complex eigenvalues occur in complex conjugate pairs.

2.2 Basic geometry of complex numbers

Having at our disposal the fact that complex numbers are literally the *vectors* from $\mathbf{R}^2$ (or sometimes it is more convenient to look at them as the *points* in the plane), we start with very basic terminology, illustrating it with geometric pictures.

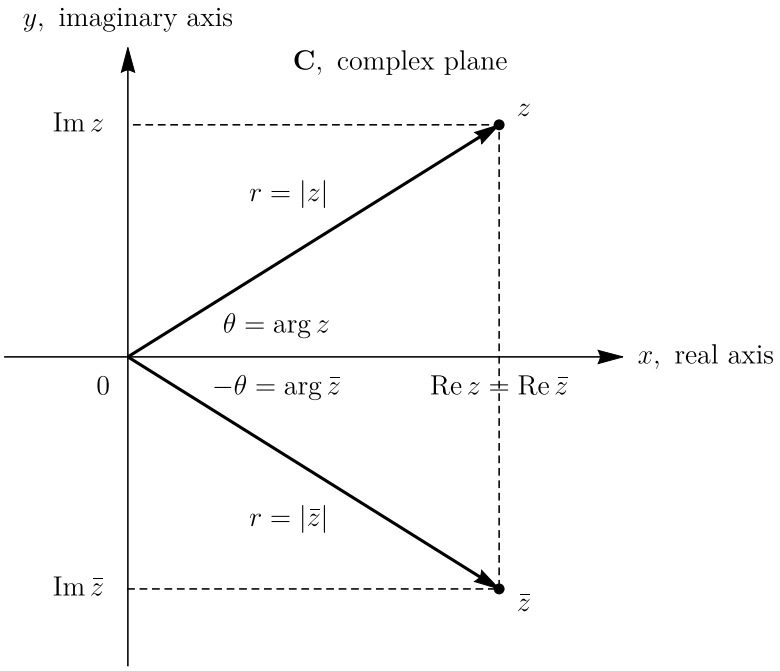

Figure 1: Complex plane.

For each complex number $z = x + \mathrm{i}y$ we have a vector $[x \;\; y]^\top$ and/or a point with (rectangular or Cartesian) coordinates (x, y). The coordinate x is called the real part of z, and y is called the imaginary part of y, the corresponding notations were introduced above. Note that geometrically

conjugation is simply a reflection with respect to the x-axis, which is, for obvious reasons, is called the real axis (the y-axis is called the imaginary axis for the same reasons). The whole plane $\mathbf{R}^2$ is called the complex plane $\mathbf{C}$. The complex numbers on the real axis are identified with usual real numbers, and the numbers on the imaginary axis are called purely imaginary.

Since $z \in \mathbf{C}$ is geometrically a vector (see the figure), we can always calculate its length $|z|$, which is given, by Pythagoras theorem,

$$|z| = |x + \mathrm{i}y| = \sqrt{x^2 + y^2}\,,$$

and all the square roots of real numbers in these notes should be understood as *positive* square roots if not states otherwise. The length of the vector z is naturally called its *modulus*, or *absolute value*. I hope that all the students remember that for our usual absolute value $|x|$ of a real number x it is true that $|x+y| \le |x| + |y|$ (can you prove this inequality?). It turns out the same is true for any complex $z, w \in \mathbf{C}$:

$$|z + w| \le |z| + |w|. \tag{2.1}$$

It is not difficult to prove it algebraically, but much nicer just to see it geometrically. For this note that addition of complex numbers is actually our usual addition of vectors (by a triangle or parallelogram law, see Fig. 2), and hence inequality (2.1) compares the length of one side of a triangle with vertices $z, w, z + w$ with the sum of the lengthes of two other sides, and therefore becomes obvious.

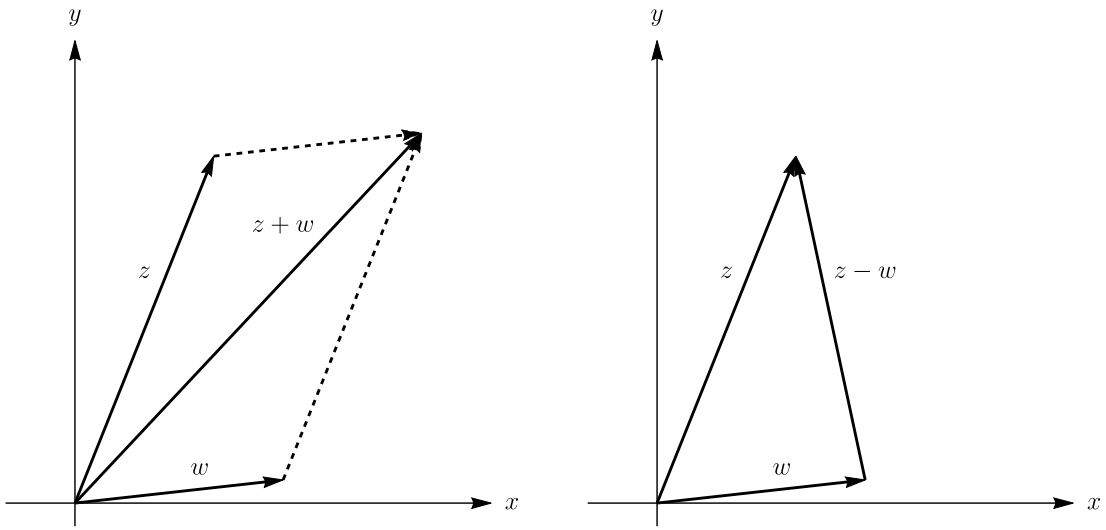

Figure 2: Geometric interpretation of complex addition and subtraction.

Problem 2.4. Prove (2.1) algebraically.

Since $z - w = z + (-w)$, and $-w$ geometrically is the vector that is obtained reflecting w with respect to the origin, then $|z-w|$ is exactly the distance between the points z and w (see Fig. 2, right and note that from now on I will use the one geometric interpretation of complex numbers, either vectors or points, which is more convenient for a given situation, without explicitly mentioning it). It is actually our usual Euclidian distance between two points and of course non-negative, symmetric and satisfies the triangle inequality $|z-w| \le |z-v| + |w-v|$ for any complex $z, v, w \in \mathbf{C}$ (why the last

inequality is true?). In more formal words complex plane $\mathbf{C}$ is a metric space with distance function $d(z,w) = |z-w|$ (even more importantly, it is a *complete* metric space, which should be proved by those who discussed completeness in other classes).

One more important remark is to note that

$$|z|^2 = x^2 + y^2 = z\bar{z},$$

which is often very helpful.

There is no straightforward geometric multiplication of complex numbers in (rectangular) coordinates (x,y), but remember that we always can pass to *polar* coordinates (r,θ) (textbook uses letter ϕ instead of θ, but there is small caveat here: Mathematicians prefer to use θ, physicists — ϕ, and it is not a big deal while we stay on the plane, when we move, however, to the three dimensional world and spherical coordinates, mathematicians call the third coordinate ϕ, physicists — θ and since they are not symmetric, it leads to different formulas, so be careful).

Recall that the polar coordinates of a point A are given by the distance r of this point from the origin and by the angle θ of the vector OA with the polar axis (if A coincides with the origin the angle is undetermined). Here is one subtle thing: polar coordinates are not defined uniquely since, e.g., angles θ and $\theta + 2\pi k$, $k \in \mathbf{Z}$ are indistinguishable. I have, using the basic trigonometry, that for the complex number $z = x + \mathrm{i}y$, its polar coordinates are

$$r = |z| = \sqrt{x^2+y^2},$$

and the corresponding angle, which is called the *argument* of the complex number z, satisfies

$$\tan\theta = \frac{y}{x}\,.$$

The argument is usually denoted $\operatorname{Arg} z$. In other words $|z|$ and $\operatorname{Arg} z$ are the polar coordinates r,θ of the point with rectangular coordinates (x,y). There is only one argument that satisfies the condition $-\pi < \theta \le \pi$, and I will call this (unique) argument of z as *principal argument*, $\theta = \arg z$ (I note that in some textbooks the notation Arg and arg is used in the opposite way, in my notes the first capital letter will mean that the result is multivalued).

Now using $\tan\theta = y/x$ I would like to determine $\arg z$. Since arctan maps any real number to the interval $(-\pi/2, \pi/2)$ I must have (make a figure if you are confused here)

$$\arg z = \begin{cases} \arctan\frac{y}{x}\,, & x > 0, \\ \arctan\frac{y}{x} + \pi, & x < 0, y \ge 0, \\ \arctan\frac{y}{x} - \pi, & x < 0, y < 0, \\ \frac{\pi}{2}\,, & x = 0, y > 0, \\ -\frac{\pi}{2}\,, & x = 0, y < 0, \\ \text{indeterminate}, & x = y = 0. \end{cases}$$

We of course have (see Fig. 1) $\arg z = -\arg\bar{z}$ (is it true for any z?).

To go back from polar coordinates to rectangular, one of course uses

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and hence

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = |z|(\cos \operatorname{Arg} z + \mathrm{i}\sin \operatorname{Arg} z).$$

The last equality is the *polar* form of a complex number z.

Finally I am ready to see the geometric meaning of complex multiplication. Let z_1, z_2 be two complex numbers

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2).$$

I have

$$\begin{aligned} z_1 z_2 &= r_1 r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)), \end{aligned}$$

in other words

$$|z_1 z_2| = |z_1||z_2|, \quad \operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where the last equality means that the set of all possible values $\operatorname{Arg} z_1 z_2$ is obtained by forming all possible sums from the sets $\operatorname{Arg} z_1$ and $\operatorname{Arg} z_2$. Ones again in words: to multiply two complex numbers means geometrically to obtain the vector with the modulus with is the product of two moduli, and with the argument which is the sum of the corresponding arguments (make a figure for, say, $z = 1$ and $w = i$).

Since

$$z^{-1} = \frac{\bar z}{|z|^2} \implies \frac{w}{z} = wz^{-1},$$

and the modulus of $|z^{-1}| = 1/|z|$ and argument $\operatorname{Arg} z^{-1} = -\operatorname{Arg} z$, where the last equality understood modulo 2π, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2\,.$$

2.3 Powers and roots

Since for any natural n

$$z^n = z \cdot \ldots \cdot z$$

and using the formula for complex multiplication in polar form derived above, I have

$$z^n = \big(r(\cos\theta + \mathrm{i}\sin\theta)\big)^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta).$$

If $r = 1$ then I obtain *De Moivre's theorem*

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

If I define $z^0 = 1$ and $z^{-n} = \frac{1}{z^n}$ then De Moivre's theorem becomes true for any integer $n \in \mathbf{Z}$. I will use this formula to solve the equation

$$z^n = 1,$$

in other words I want to determine all possible n-th roots of unity $\sqrt[n]{1}$.

Since $|1| = 1$ and $\operatorname{Arg} 1 = 2\pi k$ for $k \in \mathbf{Z}$, I have, assuming that $z = r(\cos\theta + \mathrm{i}\sin\theta)$,

$$r^n(\cos n\theta + \mathrm{i}\sin n\theta) = \cos 2\pi k + \mathrm{i}\sin 2\pi k,$$

which implies that $r^n = 1 \implies r = 1$, and

$$\theta = \frac{2\pi k}{n}, \quad k \in \mathbf{Z}.$$

How many really different θ did I get? Note that for $k = 0, 1, \ldots, n-1$ all θ will be between 0 and 2π, but for $k = n$ I get $\theta = 2\pi$, which corresponds to the same polar angle as $\theta = 0$. Therefore I obtained

Proposition 2.2. *Equation $z^n = 1$ has always n distinct roots*

$$\hat{z}_k = \sqrt[n]{1} = \cos\frac{2\pi k}{n} + \mathrm{i}\sin\frac{2\pi k}{n}, \quad k = 0, 1, \ldots, n-1,$$

which are called the n-th roots of unity.

In exactly the same way I can solve the equation

$$z^n = w$$

for a given complex $w = \rho(\cos\varphi + \mathrm{i}\sin\varphi)$.

Fill in the missed details to convince yourself that in this case

$$z_k = \sqrt[n]{w} = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2\pi k}{n} + \mathrm{i}\sin\frac{\varphi + 2\pi k}{n}\right), \quad k = 0, \ldots, n-1,$$

where φ is any argument of w. Using the formula for the multiplication of complex numbers it follows that I can write

$$z_k = \sqrt[n]{w} = \sqrt[n]{\rho}\left(\cos\frac{\arg w}{n} + \sin\frac{\arg w}{n}\right)\hat{z}_k, \quad k = 0, \ldots, n-1.$$

Problem 2.5. Show that in general the equalities

$$\arg zw = \arg z + \arg w, \quad \arg\frac{z}{w} = \arg z - \arg w$$

are incorrect.

Problem 2.6. Show that the sum of all the n-th roots of unity is always zero ($n \neq 1$). What geometric fact does it express?

Problem 2.7. In the first lecture I have derived Cardano's formula to solve a depressed cubic polynomial. I told you that this formula allows to find one *real* root of it, and two others can be found from the corresponding quadratic equation. Actually, using the roots of unity the formula that I derived can be modified to give all three (in general complex) roots. Can you fill in the necessary details?

2.4 Examples with solutions

Let us see how the material I discussed so far in this lecture can be used to solve concrete examples.

Example 2.3. What is the polar form of $z = -1 - \mathrm{i}\sqrt{3}$? I have

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2, \quad \tan\theta = \tan\arg z = \frac{-\sqrt{3}}{-1} = \sqrt{3} \implies \arg z = -\frac{2\pi}{3},$$

hence

$$-1 - \mathrm{i}\sqrt{3} = 2\left(\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right).$$

Example 2.4. What is the modulus and argument of

$$z = -\sin\frac{\pi}{8} - \mathrm{i}\cos\frac{\pi}{8}?$$

The modulus is easy since

$$|z| = \sqrt{\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}} = 1.$$

For the argument we have

$$\arg z = -\pi + \arctan\frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} = -\pi + \arctan\cot\frac{\pi}{8} = -\pi + \arctan\tan\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = -\frac{5\pi}{8},$$

hence

$$\operatorname{Arg} z = -\frac{5\pi}{8} + 2\pi k, \quad k \in \mathbf{Z}.$$

Example 2.5. Compute $(-1 + \mathrm{i}\sqrt{3})^{60}$.

First

$$-1 + \mathrm{i}\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right).$$

Hence by De Moivre's formula

$$(-1 + \mathrm{i}\sqrt{3})^{60} = 2^{60}\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)^{60} = 2^{60}\left(\cos 60\frac{2\pi}{3} + \mathrm{i}\sin 60\frac{2\pi}{3}\right) = 2^{60}.$$

Example 2.6. Find a formula for $\sin 3\theta$.

By De Moivre's formula

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta.$$

Raising to the third power the left hand side I get

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$$

and recalling that two complex numbers are equal if and only if their real and imaginary parts are equal simultaneously, I conclude

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Example 2.7. Compute all $\sqrt[4]{1 - \mathrm{i}}$. I have

$$1 - \mathrm{i} = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right),$$

hence by the formula I have I get

$$\sqrt[4]{1 - i} = \sqrt[8]{2}\left(\cos\frac{-\pi/4 + 2\pi k}{4} + \mathrm{i}\sin\frac{-\pi/4 + 2\pi k}{4}\right), \quad k = 0, 1, 2, 3.$$